CSM – 52/17
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

*Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and **three** of the remaining questions selecting at least **one** from each Section.*

SECTION – A

1. Answer any **five** of the following :

(a) Prove that every subgroup of a cyclic group is itself a cyclic group. 12

(b) Let S = {(x, y, z) / x + y + z = 0}, x, y, z being real. Prove that S is a subspace of R^3. Find a basis of S. 12

(c) State Cayley-Hamilton theorem and use it to calculate the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$ 12

(d) A variable plane is at a constant distance p from the origin O and meets the axes at A, B and C. Show that the locus of the centroid of the tetrahedron OABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}.$$ 12

(e) Show that the triangle formed by the straight lines $ax^2 + 2hxy + by^2 = 0$ and the straight line $\ell x + my = 1$ is right angled if

$(a + b)(a\ell^2 + 2h\ell m + bm^2) = 0.$ 12

(f) Prove that a finite integral domain is a field. 12

2. (a) If p be a prime and p is not a divisor of a, then $a^{p-1} \equiv 1 \pmod p$. 15

(b) If H is a cyclic normal subgroup of a group G, then show that every subgroup of H is normal in G. 15

(c) If Z is the set of integers then show that $Z\left[\sqrt{-3}\right] = \{a + b\sqrt{-3} : a, b \in Z\}$ is not a unique factorization domain. 15

(d) Let F be a field and let p(x) be an irreducible polynomial over F. Let <p(x)> be the ideal generated by p(x). Prove that <p(x)> is a maximal ideal. 15

3. (a) Find a basis for R^3 that contains the vectors (1, 2, 0) and (1, 3, 1). 15

(b) Let $T : M_{2,1} \to M_{2,3}$ be a linear transformation defined by (with usual notations) $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2 & 1 & 3\\4 & 1 & 5\end{pmatrix}$, $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}6 & 1 & 0\\0 & 0 & 2\end{pmatrix}$. Find $T\begin{pmatrix}x\\y\end{pmatrix}$. 15

(c) Obtain the normal form under congruence and find the rank and signature of the symmetric matrix $\begin{pmatrix}2 & 4 & 3\\4 & 6 & 3\\3 & 3 & 1\end{pmatrix}$. 15

(d) Show that the eigen values of a real symmetric matrix are all real. 15

4. (a) If a point lies on the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$, prove that its polar with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, touches the ellipse

$$\frac{a'^2}{a^4}x^2+\frac{b'^2}{b^4}y^2=1.$$ 15

(b) Prove that the straight line $\ell x + my + n = 0$ touches the parabola $y^2 - 4ax + 4ab = 0$ if $\ell^2 b + \ell n = am^2$. 15

(c) Find the equation of the plane passing through the straight line $3x + y + 2z - 7 = 0 = x + y - z + 4$ and perpendicular to the plane $2x + y + z = 5$. 15

(d) Show that the angle between the lines of section of the plane $3x + y + 5z = 0$ and the cone $6yz - 2zx + 5xy = 0$ is $\cos^{-1}\left(\frac{1}{6}\right)$. 15

SECTION – B

5. Answer any **five** of the following :

(a) Prove that a monotone increasing sequence, if bounded above, is convergent and the sequence converges to the upper bound. 12

(b) Find Laurent series for : 12

(i) $\dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$

(ii) $\dfrac{1}{z^2(z-3)^2}$ about $z = 3$

(c) By means of contour integration, evaluate

$\int_0^\infty \dfrac{(\log_e u)^2}{u^2+1} du.$ 12

(d) Examine the convergence of the series

$1+\dfrac{x}{1!}+\dfrac{2^2x^2}{2!}+\dfrac{3^3x^3}{3!}+......, x > 0.$ 12

(e) An area bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of the tangent. Find the volume so generated. 12

(f) Calculate the curvature at the point u of the curve given by the parametric equations $x = a(3u - u^3)$, $y = 3au^2$, $z = a(3u + u^2)$. 12

6. (a) Let $f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$.

Show that f is not Riemann-integrable on [a, b]. 15

(b) If $\underset{n\to\infty}{\text{Lt}}\, a_n = \ell$, then prove that $\underset{n\to\infty}{\text{Lt}} \frac{a_1 + a_2 + \ldots + a_n}{n} = \ell$. 15

(c) If $u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find f(z) in terms of z. 15

(d) Use the method of contour integration to prove that $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2}$, $0 < a < 1$. 15

7. (a) Express $\int_0^1 x^m (1 - x^n)^p \, dx$ in terms of Gamma function and hence evaluate the integral $\int_0^1 x^6 \sqrt{1 - x^2}\, dx$. 15

(b) Discuss the applicability of Rolle's theorem on $f(x) = e^{-x}(\cos x - \sin x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

15

(c) Find the area of the region bounded by the upper half of the circle $x^2 + y^2 = 25$, the x-axis and the ordinates $x = -3$ and $x = 4$.

15

(d) Find the surface of a sphere generated by the circle $x^2 + y^2 = a^2$ about x-axis.

15

8. (a) Evaluate $\vec{\nabla} \cdot \left[r \vec{\nabla}\left(\frac{1}{r^3}\right)\right]$, where $r = \sqrt{x^2 + y^2 + z^2}$.

15

(b) Prove that if $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \vec{F} \cdot d\vec{r} = 0$ for all closed paths in the region and conversely.

15

(c) Prove that $\iiint_V \vec{\nabla} \phi \, dv = \iint_S \phi \, \vec{n} \, ds$, where $\vec{n}$ is unit normal vector to the surface S. 15

(d) Using Green's theorem in the plane, show that the area bounded by a simple closed curve C is given by $\frac{1}{2} \oint_C (xdy - ydx)$. Hence, find the area of the ellipse $x = a \cos\theta$, $y = b \sin\theta$. 15